

Dworkin's Paradox

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Abstract

How to distribute welfare in a society is a key issue in the subject of distributional justice, which is deeply involved with notions of fairness. Following a thought experiment by Dworkin, this work considers a society of individuals with different preferences on the welfare distribution and an official to mediate the coordination among them. Based on a simple assumption that an individual's welfare is proportional to how her preference is fulfilled by the actual distribution, we show that an egalitarian preference is a strict Nash equilibrium and can be favorable even in certain inhomogeneous situations. These suggest how communication can encourage and secure a notion of fairness.

Introduction

Background

The concept of distributive justice has been extensively studied in political philosophy and economics over the past few decades. One of the most important milestones in this field is Rawls' *A Theory of Justice* [1], which put forward equality as an outcome of the social contract that individuals behind a veil of ignorance should agree on. Even though egalitarian doctrine that all human persons are equal in fundamental worth or moral status is a commonly shared idea, egalitarianism turns out to be a contested concept. There have been several divergent understandings of the meaning of equality, ways to achieve equality, or the metric to measure equality [2, 3]. For instance, someone who puts more emphasis on equality of opportunity may have a very different opinion from those who put emphasis on equality of incomes in spite of the overall agreement on the concept of equality per se.

Among numerous dimensions where egalitarianism varies, the question of what should be equalized is one that many different theories are competing on; is it opportunity, capabilities, resource or welfare [2–6]? Dworkin, one of the most influential proponents of resource egalitarianism, admits the immediate appeal of the idea that it must ultimately be equality of welfare insofar as equality is important, and examines the logical consistency and practical applicability of this welfare egalitarianism [7]. According to Dworkin, welfare egalitarianism, concerned with equality in every person's overall satisfaction, has an inconsistency in its logic. For example, if one accepts the idea that those who are handicapped need more resources to achieve equal welfare, the same argument should apply to those who have expensive tastes for the same reason. However, one should immediately recognize that the appeal of welfare egalitarianism becomes much less strong in the case of expensive tastes than in the case of the handicapped. The fact that the same idea can be accepted in some cases and seems disturbing in other cases reveals a logical inconsistency of welfare egalitarianism. Dworkin also criticized welfare-based egalitarianism on that it inevitably relies on the possibility of interpersonal comparisons of utility, which places a large burden to a policy maker in practice. Lastly, Dworkin argues that it would probably prove impossible to reach a reasonable degree of equality in this conception in a community whose members held very different and very deeply felt political theories about justice in distribution.

The last point is the one that we focus on in this paper. We will show that the existence of contradictory political theories does not immediately lead to the impossibility but can be formulated as dynamics which admits a unique solution under certain assumptions. By doing this, we will argue that the reasoning in [7] can be regarded as a tool to analyze and advocate the idea of equality in welfare. To some extent, this is complementary to a previous work which argues that one can reach the idea of equality in welfare by starting from that of equality in resources [8]. Following the logic that Dworkin used when he showed the impossibility to reach an agreement on redistribution in terms of welfare, we also set aside the issues of logical inconsistency and inter-personal comparisons of individuals' welfare. To focus on the relationship between individual preferences and resulting welfare, we also set aside the issue of impartiality (see, e.g., [9, 10]). We will further assume that individuals have preferences over the distribution of welfare among them [11, 12]. Many theoretical and experimental studies have shown that people are concerned with equality and fairness and often persist in fairness even when they lose monetary payoffs in doing so, e.g., in the public games, ultimatum games or dictator games. This behavior cannot be explained based on the assumption of self-regarding preferences but of others-regarding preferences or social preferences. Our formulation in this paper can serve as a systematic description for such an approach, because it tells us a way to translate others' payoffs into an individual's with respect to individual preference.

Model

Let the concept of welfare be understood as the fulfillment of preferences [13], including success in political preferences, i.e., opinions of how welfare should be distributed. Then the reasoning by Dworkin [7] argues that such an egalitarian society, where everyone is concerned with the equality, will end up with supporting non-egalitarians by its own logic. Suppose that a bigot enters an egalitarian society, with an opinion that some people deserve more than the others. This person will feel frustrated to see that her political preferences are not accepted by egalitarian neighbors, and her welfare becomes relatively lower than the others'. If there is an official committed to compensating for inequality in welfare, by reallocating resources for example, the bigot should get extra resources from the official due to her political frustration, *because* she does not support the egalitarian idea of the society. This is called Dworkin's paradox in this work. Particularly we note that it can serve as an idealized model to represent our understanding of a modern democratic society. We will look into this hypothetical society a little closer.

Imagine a society of N persons and an official. The official, representing a social institution, exists to mediate the global coordination. We assume that the total amount of welfare to be distributed among the persons is fixed as unity, and that the welfare is infinitely divisible, since we are interested only in relative fractions rather than absolute amounts that individuals have. The official herself does not take part in sharing the welfare, but only receives the N persons' opinions and find a way to distribute the welfare among them. Let each person i have a certain preference about how the welfare should be distributed, say $\mathbf{v}_i = (v_{i1}, v_{i2}, \dots, v_{iN})$ with $\sum_j v_{ij} = 1$. We denote the actual welfare distribution as $\mathbf{r} = (r_1, r_2, \dots, r_N)$. The person i 's welfare is determined by the extent to which her preference is fulfilled. In other words, we consider an equation

$$r_i = \mathcal{F}(\mathbf{v}_i, \mathbf{r}) \quad (1)$$

with a certain function \mathcal{F} , which is assumed to equally apply to all the persons. We suppose that the official wants to announce a stable welfare distribution \mathbf{r} such that each person's relative share remains unchanged after the announcement, which means that \mathbf{r} solves Eq. (1) self-consistently. It is important to note that the preferences reported by each individual are assumed to be true and available to the official at every moment, which helps us to focus on basic ideas of the paradox. A few remarks are in order. First, we emphasize that the official plays only a passive role in this setup. As we will see below, the society reaches the same self-consistent solution as long as every person's welfare becomes public knowledge all the time. The official may guarantee such information to be accessible and accelerate the coordination

but the official is basically assigned limited tasks compared to the original argument. Second, related to the first point, we do not require the division of welfare to be impartial from a certain observer's point of view. Our question is simply how much fulfillment one can get depending on her preference. In this sense, our approach differs from the impartial-division problem [9] and does not touch conceptual difficulties of impartiality (see, e.g., [10]). Finally, individuals are not behind the veil of ignorance. Rather, each of them is supposed to construct a concrete opinion about every other individual using any kind of available information. Although this can impose practical difficulties in a large society, it helps us avoid any theoretical ambiguity or conflict with the ethic of priority [14] found in the veil of ignorance [15].

In order to give a more concrete form to Eq. (1), we first consider how to measure similarity or affinity between distributions and then plug it into Eq. (1). Suppose two arbitrary distributions, $\mathbf{p} = (p_1, \dots, p_N)$ and $\mathbf{q} = (q_1, \dots, q_N)$, with $p_i \geq 0$, $q_i \geq 0$, and $\sum_i^N p_i = \sum_i^N q_i = 1$. We define a suitable affinity function $\rho_N(\mathbf{p}, \mathbf{q})$ between them, whose specific functional form will be characterized by requiring the following four postulates [16]. First, we postulate separability, which means that one can refine affinity contribution from a certain bin by looking into the bin in a higher resolution without referring to the outside of the bin. Second, we postulate invariance under permutation, because every bin is equivalent. Third, the affinity should be non-negative, i.e., $\rho_N(\mathbf{p}, \mathbf{q}) \geq 0$, where $\rho_N(\mathbf{p}, \mathbf{q}) = 0$ if and only if \mathbf{p} is orthogonal to \mathbf{q} , whereas a maximum value is obtained if and only if $\mathbf{p} = \mathbf{q}$. The distributions \mathbf{p} and \mathbf{q} are orthogonal when $p_i = 0$ for every non-zero q_i and vice versa. Last, it should be symmetric in the sense that $\rho_N(\mathbf{p}, \mathbf{q}) = \rho_N(\mathbf{q}, \mathbf{p})$, which is intuitively justified. These four postulates characterize our affinity function as

$$\rho_N(\mathbf{p}, \mathbf{q}) \propto \sum_{i=1}^N (p_i q_i)^{1/2}, \quad (2)$$

commonly known as the Bhattacharyya measure [17]. While details of the derivation are shown in Appendix, this functional form has a clear geometric interpretation: it can be viewed as a dot product of two vectors $(\sqrt{x_1}, \sqrt{x_2}, \dots, \sqrt{x_N})$ and $(\sqrt{y_1}, \sqrt{y_2}, \dots, \sqrt{y_N})$, both of which are located on an N -dimensional unit sphere by $\sum_i (\sqrt{x_i})^2 = \sum_i (\sqrt{y_i})^2 = 1$. It therefore becomes maximized when two vectors point in the same direction.

It is plausible to assume that the function $\mathcal{F}(\mathbf{v}_i, \mathbf{r})$ in Eq. (1) will be a non-decreasing function of the affinity between \mathbf{v}_i and \mathbf{r} , so that $\mathcal{F}(\mathbf{v}_i, \mathbf{r}) = \mathcal{F}[\rho_N(\mathbf{v}_i, \mathbf{r})]$. Specifically, we infer that $\mathcal{F}(\mathbf{v}_i, \mathbf{r}) \propto \rho_N^2(\mathbf{v}_i, \mathbf{r})$, since $\rho_N(\mathbf{v}_i, \mathbf{r})$ contains dimensionality of $\sqrt{r_i}$ according to Eq. (2). The precise value of the proportionality coefficient should be determined by the normalization condition of \mathbf{r} . For notational convenience, let us define $\mathbf{s} \equiv (s_1, s_2, \dots, s_N)$ with $s_i \equiv \sqrt{r_i}$ and $\mathbf{w}_i \equiv (w_{i1}, w_{i2}, \dots, w_{iN})$ with $w_{ij} \equiv \sqrt{v_{ij}}$. Equation (1) then leads to $s_i \propto \sum_j w_{ij} s_j$, or in a matrix form,

$$\mathbf{s} = \lambda^{-1} \mathcal{W} \mathbf{s} \quad (3)$$

where $\mathcal{W} \equiv \{w_{ij}\}$ and λ is for normalizing $|\mathbf{s}|^2$, the total welfare. This formalism is reminiscent of the quantum mechanics, where a wavefunction ψ is obtained by solving an eigenvalue problem $\mathcal{H}\psi = E\psi$ with a Hamiltonian matrix \mathcal{H} and its eigenvalue E . What one can measure in experiments is probability density $|\psi|^2$. An N -dimensional matrix preserving $|\mathbf{s}|^2$ is called orthogonal, and its degrees of freedom is the number of possible planes of rotation in N dimension, which is $N(N-1)/2$. Since \mathcal{W} generally has N^2 elements and N normalization conditions, it has $N(N-1)$ degrees of freedom, so the magnitude of λ will differ from one in general.

Results

Two-person case

The simplest example of \mathcal{W} describes a situation where two persons have not ever conceived of each other as a society member to share welfare with. The corresponding matrix is written as

$$\mathcal{W} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

This identity matrix does not change the input state at all, which means that the official cannot really coordinate these two indifferent persons' opinions in the way that we have assumed. This is actually an example of a reducible matrix [18] or a society that can be divided into smaller pieces: \mathcal{W} is irreducible if there exists a sequence of $[k_1, k_2, \dots, k_n]$ for any i and j such that $w_{ik_1} \times w_{k_1 k_2} \times \dots \times w_{k_n j}$ is non-zero. Otherwise, \mathcal{W} is reducible. Such a reducible case is not our concern since a society is meaningful only when individuals interact with each other. Henceforth, only irreducible cases are considered. Then, unless everyone has zero self-interest, one can prove that there exists a unique stable distribution \mathbf{r}_p for every \mathcal{W} by using the Perron-Frobenius theorem [18]. In other words, \mathbf{r}_p is the only stable fixed point under the action of \mathcal{W} , so the official should distribute welfare as given by \mathbf{r}_p .

Let us consider a situation where an egalitarian with $\mathbf{v}_1 = (1/2, 1/2)$ meets a selfish person with $\mathbf{v}_2 = (0.01, 0.99)$. The corresponding matrix formulation will be

$$\mathcal{W} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/10 & \sqrt{99}/10 \end{pmatrix}. \quad (4)$$

and its stable welfare distribution is obtained by analyzing eigenvectors as $\mathbf{r}_p \approx (0.72, 0.28)$. We arrive at this \mathbf{r}_p even if we start from $\mathbf{r} = (0.5, 0.5)$ for the following reason: let \mathbf{r} be known to both the persons every time step. The egalitarian first feels happy to see the initial equality in $\mathbf{r} = (0.5, 0.5)$, while the selfish person feels unsatisfied, which makes a difference at the next step. The drop in r_2 makes the selfish person even more upset, so her welfare continues to decrease until it reaches the stable value, $r_2 = 0.28$.

We now show that egalitarianism is the minimax solution of this two-person zero-sum game [19]. Generalizing Eq. (4) as

$$\mathcal{W} = \begin{pmatrix} \frac{\sqrt{v_{11}}}{\sqrt{1-v_{22}}} & \frac{\sqrt{1-v_{11}}}{\sqrt{v_{22}}} \end{pmatrix}, \quad (5)$$

the eigenvalue analysis yields the converged share for the first person, r_1 , as shown in Fig. 1. It is a saddle-like shape and this person can minimize risk when she has demanded a moderate share of $1/2$ at the first place. The same is true for the other person as well. Although we have assumed fixed preferences in developing the model, if the preferences can evolve in the long run to maximize individual welfare, therefore, this plot shows that this two-person case will lead to an equal welfare distribution.

In practice, a selfish person can be tempted to deceive the official by reporting a false preference to receive a larger share. Provided that person 2 has claimed her self-interest as a certain value v_{22} , person 1 can always compute the best reply $\tilde{v}_{11} = b(v_{22})$ by looking up the maximal share \tilde{r}_1 at the given v_{22} in Fig. 1. Even if her true self-interest v_{11} is higher than this false \tilde{v}_{11} , she should still report \tilde{v}_{11} to the official, knowing that she cannot get better than \tilde{r}_1 in any way. When person 1 has chosen \tilde{v}_{11} for this reason, the same consideration will lead person 2 to choose $\tilde{v}_{22} = b(\tilde{v}_{11}) = b[b(v_{22})]$, and this reasoning can be repeated between them *ad infinitum*. Such a strategic consideration eventually forces them to choose the egalitarian preference in common, since successive iteration of the best-reply function b drives every initial input $v_{22} \in [0, 1)$ into the egalitarian fixed point, although none of the players are really egalitarians.

Egalitarianism as a Nash equilibrium

Let us consider an N -person case where all except one are egalitarian. That is, $\mathbf{v}_i = (1/N, \dots, 1/N)$ for every $i \neq 1$. We observe that these $N - 1$ persons will have exactly the same welfare since they always get the same amount of affinity for any welfare distribution \mathbf{r} . Let us thus denote every egalitarian's welfare as a single variable R . Recalling the separability, we find that all the elements v_{1i} with $i \neq 1$ must be the same in order to maximize person 1's welfare, because her preference about the egalitarians should match with welfare distribution among them. Therefore, person 1 should have a preference of $\mathbf{v}_1 = (v_{11}, V, V, \dots, V)$ with $V \equiv (1 - v_{11})/(N - 1)$. As a consequence, the full $N \times N$ matrix calculation can be simplified to the following 2×2 matrix calculation

$$\begin{aligned} & \begin{pmatrix} \sqrt{v_{11}} & (N-1)\sqrt{V} \\ \sqrt{1/N} & (N-1)\sqrt{1/N} \end{pmatrix} \begin{pmatrix} \sqrt{r_1} \\ \sqrt{R} \end{pmatrix} \\ &= \begin{pmatrix} \sqrt{v_{11}r_1} + (N-1)\sqrt{VR} \\ \sqrt{r_1/N} + (N-1)\sqrt{R/N} \end{pmatrix} \end{aligned}$$

with a normalization condition $r_1 + (N - 1)R = 1$. The stable distribution from the simplified matrix then yields r_1 as a function of v_{11} , which has $\partial r_1 / \partial v_{11} = 0$ and $r_1 = 1/N$ at $v_{11} = 1/N$. In short, the best possible preference for person 1 is an egalitarian one. If egalitarianism is pervasive, one gets worse off by having another type of preference, which means that egalitarianism is a strict Nash equilibrium [20]. This reproduces Dworkin's paradox in mathematical terms in the sense that a non-egalitarian in an egalitarian society will have relatively less welfare. A difference from the original paradox is that the official cannot really compensate the non-egalitarian within our formulation since the welfare distribution will converge to the same point again as soon as the compensation is known in public.

Inhomogeneous society

Egalitarian preference can be still favorable even when people are all different. For instance, people are not equally born. Let this unavoidable inequality be described by a uniform random variable $\zeta_i \in [-1, 1]$. Person i 's overall political preference can be described by another uniform random variable $\phi_i \in (-1/N, 1/N)$: for $\phi_i = 0$, this person is an egalitarian. If $\phi_i > 0$, she believes that the better deserve more, while $\phi_i < 0$ means the opposite. In addition, ϕ_i is assumed to be uncorrelated with ζ_i . The political preference is then assigned as $v_{ij} = \phi_i \zeta_j + 1/N$, which satisfies $v_{ij} > 0$. Since ζ_j is a relative quantity, one can always subtract an offset value to make $\sum_j \zeta_j = 0$, by which the normalization condition $\sum_j v_{ij} = 1$ is satisfied. We can obtain the stable distribution by taking N as a very large constant and assuming that r_i is a function of ϕ_i only. By replacing the summation in $\sqrt{r_i(\phi_i)} \propto \sum_{j=1}^N \sqrt{\phi_i \zeta_j + 1/N} \sqrt{r_j(\phi_j)}$ by an integral, we get

$$r_i(\phi_i) = 2\phi_i^{-2} \left[(\phi_i + 1/N)^{3/2} - (-\phi_i + 1/N)^{3/2} \right]^2 / (3\pi + 8),$$

where the proportionality coefficient is determined by the normalization condition. This r_i is an even function of ϕ_i with a maximum $r_i(\phi_i = 0) = 18/(3\pi + 8) \times 1/N \approx 1.033/N$, implying the highest fulfillment for an egalitarian. One could point out that this $r_i(\phi_i)$ describes just one possible distribution, not necessarily the stable one. However, we see that the integral is positive for all ϕ_i , and thus for all i , and the Perron-Frobenius theorem tells us that this positivity is true only for the stable distribution [18]. This justifies our starting assumption that person i 's welfare r_i is not determined by her innate part ζ_i but by her political preference ϕ_i in this society. It is notable that critics have said that the idea of equality in welfare is insensitive to individual responsibility [7]. As explained in [21], one may consider a set of variables characterizing an individual and classify them into two categories: the first category consists of innate properties such as talents that an individual is hardly responsible for. The second category, on the

other hand, includes choices and even some of preferences that we can connect to individual responsibility. If we regard ζ_i as representing the first category while ϕ_i as representing the second category, this example shows that each individual does take responsibility for her political preference but not for her talents. It could be also argued that the limit $N \rightarrow \infty$ squeezes $\phi_i \in (-1/N, 1/N)$ into zero so that the whole problem reduces to the egalitarian society above, where everyone gets $r_i = 1/N$. That can be regarded as a first-order approximation of this problem. The calculation given here shows that an egalitarian indeed receives 3.3% more than in the crude approximation.

Homogeneously unequal preference

In all the cases considered so far, preferences could be said to be neutral on the society level in the sense that there is no systematic bias over the whole society. Let us now imagine that $N - 1$ persons with identical preferences, $\mathbf{h} = (h_1, h_2, \dots, h_N)$, but not necessarily egalitarians. The other person indexed by k has another type of preference, $\mathbf{v}_k = (v_{k1}, v_{k2}, \dots, v_{kN})$. By the similar reasoning as in the egalitarian society, the situation can be simplified to the following 2×2 matrix:

$$\begin{pmatrix} \sqrt{v_{kk}} & \sqrt{(N-1)(1-v_{kk})} \\ \sqrt{h_k} & \sum_i'' \sqrt{h_i} \end{pmatrix},$$

where \sum_i'' means a summation over i excluding k . Recall that the $(N - 1)$ persons with an identical preference have the same amount of fulfillment so person k should not distinguish them in order to maximize affinity between her preference and the welfare distribution among them. The above matrix means that we should only determine how to divide welfare between the person k and the other $(N - 1)$ persons. The eigenvalue analysis leads to

$$r_k = (1 - v_{kk})(1 - v_{kk} + X^2/4) \quad (6)$$

with

$$X \equiv \sqrt{v_{kk}} - H - \sqrt{(\sqrt{v_{kk}} - H)^2 + 4\sqrt{h_k(N-1)(1-v_{kk})}}$$

and $H \equiv \sum_i'' \sqrt{h_i}$. We can differentiate Eq. (6) with respect to v_{kk} to find the maximum. An easier alternative way is to observe from the separability that the maximum value is obtained when $r_k = v_{kk}$, because the question is how to match person k 's preference $(v_{kk}, 1 - v_{kk})$ with the welfare distribution $(r_k, 1 - r_k)$, where the second elements represent the whole $(N - 1)$ persons. By solving $r_k = v_{kk}$ with Eq. (6), one can get v_{kk} maximizing r_k as a solution of the following equation,

$$[h_k(N-1) + H^2]w^4 - 2Hw^3 + (1 - H^2)w^2 + 2Hw - 1 = 0, \quad (7)$$

with $w \equiv \sqrt{v_{kk}}$. For the egalitarian \mathbf{h} with $h_k = 1/N$ and $H = (N-1)\sqrt{1/N}$, substituting $v_{kk} = 1/N$ satisfies Eq. (7), consistently with the analysis of the $(N - 1)$ egalitarians. We may also suppose that \mathbf{h} describes an unequal distribution so that the whole society is biased in a certain way. As a specific example, let us assume that $h_i \propto i^2$, that is, almost everyone wants people with higher indices to have more welfare. With a normalization constant, it should mean that $h_i = i^2/Z$ with $Z \equiv N(N+1)(2N+1)/6$, and we thus have

$$H = \sum_{i=1}^N \sqrt{h_i} - \sqrt{h_k} = \sqrt{6N(N+1)/(2\sqrt{2N+1})} - k/\sqrt{Z}.$$

Inserting this into Eq. (6), we plot r_k in Fig. 2, where the maximum is found at the crossing with $r_k = v_{kk}$. It is a little higher than $1/N$ for every k . We can do the same calculation for a more severe situation of inequality by setting $h_i \propto i^4$, which again yields the same conclusion with a bit larger v_{kk} . The difference between the optimal v_{kk} and $1/N$ does not vanish as $N \rightarrow \infty$ whether $h_i \propto i^2$ or i^4 . This can

be shown by inserting $v_{kk} = 1/N$ on the left-hand side of Eq. (7) and taking $N \rightarrow \infty$, which does not yield zero on the right-hand side. Therefore, the person k should demand a little more for herself than before and distribute the remainder of the preferences equally to the others, even though they are far from egalitarians, in order to get the maximum welfare.

Transient behavior

In order to see whether egalitarians can eventually take over the society, we need to check whether the egalitarian preference remains as an attractive alternative when the society has both egalitarians and non-egalitarians with significant numbers. Let us imagine an inhomogeneous society where there are roughly two large groups: every person in one group of size $N - M$ occupies a high index i and believes that the welfare should be proportional to i^2 . On the other hand, every person in the other group of size $M - 1$ has a low index and an egalitarian preference. Our question is what kind of preference is good for a person on the border, i.e., with index $i = M$. Again, since people with identical preferences will get the same amount of welfare, the focal person on the border need not distinguish the members in each group: suppose that she wishes v_{M1} for each member in the egalitarian group and v_{MN} for each member in the non-egalitarian group. The normalization condition then determines her self-interest $v_{MM} = 1 - (M - 1)v_{M1} - (N - M)v_{MN}$. Depending on how she decides v_{M1} and v_{MN} , her final welfare r_M will be calculated by analyzing the following 3×3 matrix,

$$\begin{pmatrix} (M-1)/\sqrt{N} & \sqrt{1/N} & (N-M)/\sqrt{N} \\ (M-1)\sqrt{v_{M1}} & \sqrt{v_{MM}} & (N-M)\sqrt{v_{MN}} \\ \frac{M(M-1)}{2\sqrt{Z}} & M/\sqrt{Z} & \frac{N(N+1)}{2\sqrt{Z}} - \frac{M(M+1)}{2\sqrt{Z}} \end{pmatrix}, \quad (8)$$

where the second row describes this focal person M . The conservation of the total welfare is imposed by setting $(M-1)r_{M1} + r_{MM} + (N-M)r_{MN} = 1$. When M is small, the maximum of r_M is close to the egalitarian solution $(v_{M1}, v_{MN}) = (1/N, 1/N)$ (Fig. 3A). It agrees with the result of the homogeneously unequal preferences given above since the egalitarian preference is still an absolute minority. Hence, if this person M can choose her own preference, the society will possibly have one more egalitarian. As M becomes larger, however, the situation gets different in that the maximum is located far from the egalitarian solution (Fig. 3B). It implies that the transition process toward the egalitarian direction may exhibit transient behavior, instead of being smooth all the time.

Discussion

The theory of welfare is not an empty ideal as claimed in [7]: dealing with a society where everyone has an identical non-egalitarian preference, we have found that the theory recommends something very similar to an egalitarian preference, instead of just rubber-stamping the dominant non-egalitarian opinion. In addition, this finding shows that the egalitarian society is in fact the *only* strict Nash equilibrium. We therefore conclude that our analysis gives a strong support to equality of welfare by specifying which social and political conditions make it possible.

On the other hand, our conclusion implies that a society can encourage egalitarianism by guaranteeing freedom of communication so that everyone can constantly express her fulfillment in public. In this respect, we can perhaps mention one of the central messages in [7] that “liberty is essential to any process in which equality is defined and secured.” In particular, we would like to put an extra emphasis on the communicative aspect of the liberty.

On a longer perspective, our results suggest an explanation of how the concept of fairness could develop at a certain moment in the history of evolution when human beings became able to construct internal expectation for the future and understand others’ minds by communication. It is also worth

stressing that our conclusion on egalitarianism as a strict Nash equilibrium under certain well-defined conditions is strong enough to open further theoretical extensions and empirical tests.

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Figure Legends

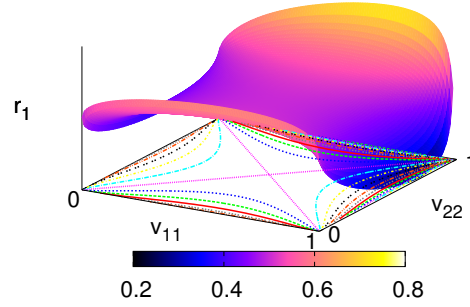


Figure 1. Person 1's welfare in the two-person case, obtained from Eq. (5). The curves on the plane show contour lines.

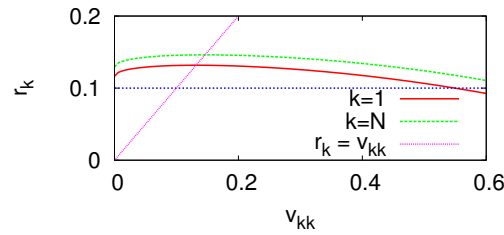


Figure 2. Equation (6) as a function of v_{kk} when $h_i \propto i^2$ for $N = 10$. The horizontal line shows $r_k = 1/N$. The maximum of r_k is located at the crossing with the line $r_k = v_{kk}$.

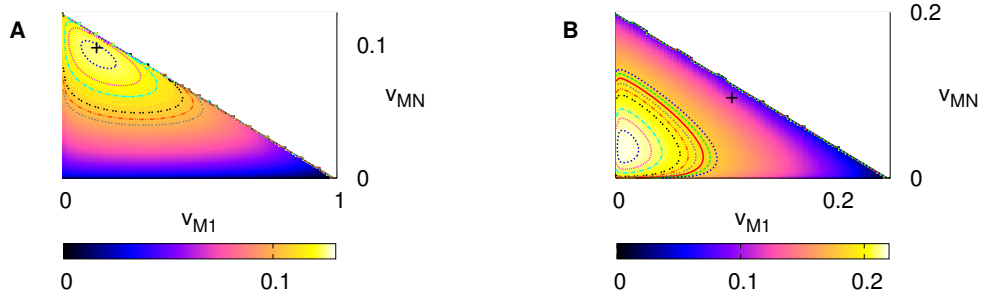


Figure 3. $r_M(v_{M1}, v_{MN})$ obtained by solving Eq. (8) within a region $0 \leq (M-1)v_{M1} + (N-M)v_{MN} \leq 1$ for $N=10$. (A) $M=2$. (B) $M=5$. The crosses show $(v_{M1}, v_{MN}) = (1/N, 1/N)$.

Appendix: derivation of the affinity function

Let us consider two arbitrary distributions, $\mathbf{p} = (p_1, \dots, p_N)$ and $\mathbf{q} = (q_1, \dots, q_N)$, with $p_i \geq 0$, $q_i \geq 0$, and $\sum_i^N p_i = \sum_i^N q_i = 1$. We will define an affinity function $\rho_N(\mathbf{p}, \mathbf{q})$ between them, whose specific functional form will be characterized by requiring the following postulates.

- P1: Separability

$$\begin{aligned} \rho_N \left(\begin{matrix} p_1, \dots, p_N \\ q_1, \dots, q_N \end{matrix} \right) &= \rho_{N-k+1} \left(\begin{matrix} P_k, p_{k+1}, \dots, p_N \\ Q_k, q_{k+1}, \dots, q_N \end{matrix} \right) \\ &+ \Delta(P_k, Q_k) \left[\rho_k \left(\begin{matrix} \frac{p_1}{P_k}, \dots, \frac{p_k}{P_k} \\ \frac{q_1}{Q_k}, \dots, \frac{q_k}{Q_k} \end{matrix} \right) - 1 \right], \end{aligned}$$

where $P_k \equiv p_1 + p_2 + \dots + p_k$ and $Q_k \equiv q_1 + q_2 + \dots + q_k$ are positive with $1 < k < N$. In addition, $\Delta(P_k, Q_k)$ is a non-negative differentiable function and converges to zero as $P_k \rightarrow 0$ or $Q_k \rightarrow 0$. The situation can be described as follows: we first observe $\mathbf{p}' = (P_k, p_{k+1}, \dots, p_N)$ and $\mathbf{q}' = (Q_k, q_{k+1}, \dots, q_N)$, instead of the real \mathbf{p} and \mathbf{q} . We can calculate affinity ρ_{N-k+1} in this resolution. The question is how affinity will change when we come to know the real \mathbf{p} and \mathbf{q} . If the substructures inside P_k and Q_k have an exact affinity by $\rho_k = 1$, nothing will change from the previous calculation, i.e., $\rho_N = \rho_{N-k+1}$. If they actually had no affinity by $\rho_k = 0$ in this better resolution, on the other hand, the overall affinity should decrease by a certain amount Δ , which will be a function of P_k and Q_k . For example, for $P_k \ll 1$ or $Q_k \ll 1$, the decrement in ρ_N will be also vanishingly small even if the subpopulations inside them look completely different.

- P2: Invariance under permutation

$$\rho_3 \left(\begin{matrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{matrix} \right) = \rho_3 \left(\begin{matrix} p_2, p_1, p_3 \\ q_2, q_1, q_3 \end{matrix} \right) = \dots = \rho_3 \left(\begin{matrix} p_3, p_2, p_1 \\ q_3, q_2, q_1 \end{matrix} \right).$$

Note that it is for $N = 3$, which is enough to prove the same property for general N in combination with the other postulates.

- P3: Non-negativity

$$\rho_N(\mathbf{p}, \mathbf{q}) \geq 0,$$

where $\rho_N(\mathbf{p}, \mathbf{q}) = 0$ if and only if \mathbf{p} is orthogonal to \mathbf{q} , whereas a maximum value is obtained if and only if $\mathbf{p} = \mathbf{q}$.

- P4: Symmetry

$$\rho_N(\mathbf{p}, \mathbf{q}) = \rho_N(\mathbf{q}, \mathbf{p}).$$

Let us explain some direct consequences of P1. For $N = 2$, it yields a trivial equality. For $N = 3$, we have

$$\rho_3 \left(\begin{matrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{matrix} \right) = \rho_2 \left(\begin{matrix} P_2, p_3 \\ Q_2, q_3 \end{matrix} \right) + \Delta(P_2, Q_2) \left[\rho_2 \left(\begin{matrix} \frac{p_1}{P_2}, \frac{p_2}{P_2} \\ \frac{q_1}{Q_2}, \frac{q_2}{Q_2} \end{matrix} \right) - 1 \right].$$

For $N = 4$, we will see two different ways to calculate affinity between (P_3, p_4) and (Q_3, q_4) . The first way is the following:

$$\begin{aligned}
& \rho_4 \left(\begin{matrix} p_1, \dots, p_4 \\ q_1, \dots, q_4 \end{matrix} \right) \\
&= \rho_2 \left(\begin{matrix} P_3, p_4 \\ Q_3, q_4 \end{matrix} \right) + \Delta(P_3, Q_3) \left[\rho_3 \left(\begin{matrix} \frac{p_1}{P_3}, \frac{p_2}{P_3}, \frac{p_3}{P_3} \\ \frac{q_1}{Q_3}, \frac{q_2}{Q_3}, \frac{q_3}{Q_3} \end{matrix} \right) - 1 \right] \\
&= \rho_2 \left(\begin{matrix} P_3, p_4 \\ Q_3, q_4 \end{matrix} \right) + \Delta(P_3, Q_3) \left\{ \rho_2 \left(\begin{matrix} \frac{P_2}{P_3}, \frac{p_3}{P_3} \\ \frac{Q_2}{Q_3}, \frac{q_3}{Q_3} \end{matrix} \right) \right. \\
&\quad \left. + \Delta \left(\frac{P_2}{P_3}, \frac{Q_2}{Q_3} \right) \left[\rho_2 \left(\begin{matrix} \frac{p_1}{P_2}, \frac{p_2}{P_2} \\ \frac{q_1}{Q_2}, \frac{q_2}{Q_2} \end{matrix} \right) - 1 \right] - 1 \right\}. \tag{9}
\end{aligned}$$

We then consider another way to calculate the same quantity as follows:

$$\begin{aligned}
& \rho_4 \left(\begin{matrix} p_1, \dots, p_4 \\ q_1, \dots, q_4 \end{matrix} \right) \\
&= \rho_3 \left(\begin{matrix} P_2, p_3, p_4 \\ Q_2, q_3, q_4 \end{matrix} \right) + \Delta(P_2, Q_2) \left[\rho_2 \left(\begin{matrix} \frac{p_1}{P_2}, \frac{p_2}{P_2} \\ \frac{q_1}{Q_2}, \frac{q_2}{Q_2} \end{matrix} \right) - 1 \right] \\
&= \rho_2 \left(\begin{matrix} P_3, p_4 \\ Q_3, q_4 \end{matrix} \right) + \Delta(P_3, Q_3) \left[\rho_2 \left(\begin{matrix} \frac{P_2}{P_3}, \frac{p_3}{P_3} \\ \frac{Q_2}{Q_3}, \frac{q_3}{Q_3} \end{matrix} \right) - 1 \right] \\
&\quad + \Delta(P_2, Q_2) \left[\rho_2 \left(\begin{matrix} \frac{p_1}{P_2}, \frac{p_2}{P_2} \\ \frac{q_1}{Q_2}, \frac{q_2}{Q_2} \end{matrix} \right) - 1 \right]. \tag{10}
\end{aligned}$$

Comparing Eq. (9) with Eq. (10), we see that

$$\Delta(P_3, Q_3) \Delta \left(\frac{P_2}{P_3}, \frac{Q_2}{Q_3} \right) = \Delta(P_2, Q_2). \tag{11}$$

Note that P_2/P_3 can take any value between zero and one, regardless of P_3 , and that Q_2/Q_3 is also independent of Q_3 in the same way, as long as P_3 and Q_3 are nonzero.

Now we define

$$g(x, y) \equiv \rho_2 \left(\begin{matrix} x, 1-x \\ y, 1-y \end{matrix} \right) - 1, \tag{12}$$

for $x, y \in [0, 1]$. From P1 and P2, it is straightforward to verify

$$g(x, y) = g(1-x, 1-y), \tag{13}$$

since

$$\begin{aligned}
& \rho_3 \left(\begin{matrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{matrix} \right) = \rho_3 \left(\begin{matrix} p_2, p_1, p_3 \\ q_2, q_1, q_3 \end{matrix} \right) \\
&= \rho_2 \left(\begin{matrix} P_2, p_3 \\ Q_2, q_3 \end{matrix} \right) + \Delta(P_2, Q_2) \left[\rho_2 \left(\begin{matrix} \frac{p_1}{P_2}, \frac{p_2}{P_2} \\ \frac{q_1}{Q_2}, \frac{q_2}{Q_2} \end{matrix} \right) - 1 \right] \\
&= \rho_2 \left(\begin{matrix} P_2, p_3 \\ Q_2, q_3 \end{matrix} \right) + \Delta(P_2, Q_2) \left[\rho_2 \left(\begin{matrix} \frac{p_2}{P_2}, \frac{p_1}{P_2} \\ \frac{q_2}{Q_2}, \frac{q_1}{Q_2} \end{matrix} \right) - 1 \right].
\end{aligned}$$

Applying P1 to another permutation

$$\rho_3 \left(\begin{matrix} p_1, p_2, p_3 \\ q_1, q_2, q_3 \end{matrix} \right) = \rho_3 \left(\begin{matrix} p_3, p_2, p_1 \\ q_3, q_2, q_1 \end{matrix} \right),$$

we are directly led to

$$\begin{aligned} & g(p_3, q_3) + \Delta(1 - p_3, 1 - q_3)g\left(\frac{p_1}{1 - p_3}, \frac{q_1}{1 - q_3}\right) \\ = & g(p_1, q_1) + \Delta(1 - p_1, 1 - q_1)g\left(\frac{p_3}{1 - p_1}, \frac{q_3}{1 - q_1}\right), \end{aligned}$$

where $p_1, p_3, q_1, q_3 \in [0, 1]$. Let us redefine the variables as $\frac{p_1}{1 - p_3} \equiv p$, $\frac{q_1}{1 - q_3} \equiv q$, $1 - p_3 \equiv r$, and $1 - q_3 \equiv s$. Each of these new variables p , q , r , and s can take an arbitrary value independently of one another, as long as $p, q \in [0, 1]$ and $r, s \in (0, 1)$. The above equality can be then rewritten as

$$\begin{aligned} & g(r, s) + \Delta(r, s)g(p, q) \\ = & g(pr, qs) + \Delta(1 - pr, 1 - qs)g\left(\frac{1 - r}{1 - pr}, \frac{1 - s}{1 - qs}\right). \end{aligned} \quad (14)$$

Defining a function

$$f(p, q, r, s) \equiv g(r, s) + [\Delta(r, s) + \Delta(1 - r, 1 - s)]g(p, q) \quad (15)$$

for $p, q, r, s \in (0, 1)$, we apply Eq. (14) to $f(p, q, r, s)$ to obtain

$$\begin{aligned} f(p, q, r, s) &= g(pr, qs) \\ &+ \Delta(1 - pr, 1 - qs)g\left(\frac{1 - r}{1 - pr}, \frac{1 - s}{1 - qs}\right) \\ &+ \Delta(1 - r, 1 - s)g(p, q). \end{aligned}$$

Note from Eq. (11) that

$$\begin{aligned} & \Delta(p_2 + p_3, q_2 + q_3)\Delta\left(\frac{p_3}{p_2 + p_3}, \frac{q_3}{q_2 + q_3}\right) \\ = & \Delta(1 - pr, 1 - qs)\Delta\left(\frac{1 - r}{1 - pr}, \frac{1 - s}{1 - qs}\right) \\ = & \Delta(1 - r, 1 - s). \end{aligned}$$

Hence, we see that

$$\begin{aligned} & f(p, q, r, s) \\ = & g(pr, qs) + \Delta(1 - pr, 1 - qs)g\left(\frac{1 - r}{1 - pr}, \frac{1 - s}{1 - qs}\right) \\ & + \Delta(1 - pr, 1 - qs)\Delta\left(\frac{1 - r}{1 - pr}, \frac{1 - s}{1 - qs}\right)g(p, q) \\ = & g(pr, qs) + \Delta(1 - pr, 1 - qs)\left[g\left(\frac{1 - r}{1 - pr}, \frac{1 - s}{1 - qs}\right) \right. \\ & \left. + \Delta\left(\frac{1 - r}{1 - pr}, \frac{1 - s}{1 - qs}\right)g(p, q)\right] \end{aligned} \quad (16)$$

$$\begin{aligned} = & g(pr, qs) + \Delta(1 - pr, 1 - qs)\left\{g\left[\frac{p(1 - r)}{1 - pr}, \frac{q(1 - s)}{1 - qs}\right] \right. \\ & \left. + \Delta\left(\frac{1 - p}{1 - qr}, \frac{1 - q}{1 - qs}\right)g(r, s)\right\}, \end{aligned} \quad (17)$$

where the last equality comes from Eq. (14). Equation (13) then gives us

$$g\left[\frac{p(1-r)}{1-pr}, \frac{q(1-s)}{1-qs}\right] = g\left(\frac{1-p}{1-pr}, \frac{1-q}{1-qs}\right),$$

relating Eqs. (16) and (17) as follows:

$$\begin{aligned} & f(p, q, r, s) \\ &= g(pr, qs) + \Delta(1-pr, 1-qs) \left[g\left(\frac{1-r}{1-pr}, \frac{1-s}{1-qs}\right) \right. \\ & \quad \left. + \Delta\left(\frac{1-r}{1-pr}, \frac{1-s}{1-qs}\right) g(p, q) \right] \\ &= g(pr, qs) + \Delta(1-pr, 1-qs) \left[g\left(\frac{1-p}{1-pr}, \frac{1-q}{1-qs}\right) \right. \\ & \quad \left. + \Delta\left(\frac{1-p}{1-pr}, \frac{1-q}{1-qs}\right) g(r, s) \right]. \end{aligned}$$

In short, we have just confirmed that

$$f(p, q, r, s) = f(r, s, p, q).$$

From the definition of $f(p, q, r, s)$ in Eq. (15), we see that

$$\frac{g(r, s)}{g(p, q)} = \frac{\Delta(r, s) + \Delta(1-r, 1-s) - 1}{\Delta(p, q) + \Delta(1-p, 1-q) - 1}.$$

For this to be true for every independent set of (p, q, r, s) , it must be that

$$g(r, s) = C[\Delta(r, s) + \Delta(1-r, 1-s) - 1]$$

with a certain constant C . Then P1 says that

$$\lim_{\substack{r \rightarrow 0 \\ s \rightarrow 1}} g(r, s) = -C,$$

which is the lowest possible value of $g(r, s)$. However, according to the definition of $g(r, s)$ in Eq. (12) and P3, this lower bound should be -1 . In other words, $C = 1$ and

$$\rho_2\left(\frac{r, 1-r}{s, 1-s}\right) = \Delta(r, s) + \Delta(1-r, 1-s).$$

We have restricted our variables as $r \in (0, 1)$ and $s \in (0, 1)$, but it is not difficult to check that the above expression can be readily used in $r \in [0, 1]$ and $s \in [0, 1]$ when Δ is characterized as will be discussed below. Then P1 works as a recursive relation, yielding an expression for general N :

$$\rho_N = \sum_{i=1}^N \Delta(p_i, q_i). \quad (18)$$

Now we ask ourselves how Δ should look. P1 and P4 imply that Δ can be expanded as $\Delta(p_i, q_i) = c_1(p_i q_i)^{\alpha_1} + c_2(p_i q_i)^{\alpha_2} + \dots$ with coefficients c_n and exponents $\alpha_n > 0$. So let us write

$$\rho_N = \sum_{i=1}^N \sum_{n=1}^{\infty} c_n (p_i q_i)^{\alpha_n}.$$

If we define $l_\alpha(\mathbf{p}, \mathbf{q}) \equiv \sum_{i=1}^N (p_i q_i)^\alpha$, it means that ρ_N should be of the following form,

$$\rho_N(\mathbf{p}, \mathbf{q}) = \sum_{n=1}^{\infty} c_n l_{\alpha_n}(\mathbf{p}, \mathbf{q}). \quad (19)$$

Among every possible l_α , it is only $l_{1/2}$ that satisfies the maximization condition in P3. This can be shown by variational calculus using a Lagrange multiplier μ ,

$$\frac{\partial}{\partial q_i} \left[l_\alpha(\mathbf{p}, \mathbf{q}) - \mu \sum_{j=1}^N q_j \right]_{q_i=p_i} = \alpha p_i^{2\alpha-1} - \mu = 0,$$

which is satisfied for any p_i when $\alpha = 1/2$ and $\mu = \alpha$. Hence we separate this term from the others in Eq. (19) as

$$\rho_N(\mathbf{p}, \mathbf{q}) = c l_{1/2}(\mathbf{p}, \mathbf{q}) + \sum'_n c_n l_{\alpha_n}(\mathbf{p}, \mathbf{q}),$$

where \sum' means that $\alpha_n = 1/2$ is excluded from the summation. There exists a maximum value in ρ_N , which is obtained by

$$\begin{aligned} \rho_N(\mathbf{p}, \mathbf{p}) &= c l_{1/2}(\mathbf{p}, \mathbf{p}) + \sum'_n c_n l_{\alpha_n}(\mathbf{p}, \mathbf{p}) \\ &= c + \sum'_n c_n \sum_i p_i^{2\alpha_n}, \end{aligned}$$

noting that $l_{1/2}(\mathbf{p}, \mathbf{p}) = \sum_i p_i = 1$. Since this value is the same for every \mathbf{p} , the last term should be kept constant for any p_i , and we have to choose $c_n = 0$ for every n in the second term. To sum up, our affinity function is characterized as

$$\rho_N(\mathbf{p}, \mathbf{q}) = c \sum_{i=1}^N (p_i q_i)^{1/2}$$

with a positive constant c , the maximum value of ρ_N .